Math 109 : Mathematical Reasoning

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Book: An Intro. to Mathematical Reasoning, Peter J. Eccles

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→ teaching → math 109
(Check Frequently)

Grading: Higher of two formulas:

(1) 20/20/20/40 (HW/MT1/MT2/Final)

(2) 20/20/60 (HW/max(MT1, MT2)/Final)

Course objectives:

* Intro to rigorous mathematics, i.e., to methods of reasoning where all assumptions are clear and conclusions follow "logically" from them.

* Emphasis on clear proof writing

Some Topics we will cover :

- Propositions, truth tables, Logical implications
- Proof techniques : - direct proof
  - proof by cases
  - proof by contradiction
  - proof by contrapositive
  - proof by induction

- Set theory
- Functions , injections, surjections, bijections
- Binomial coefficients / theorem
- Countability of sets

...

Chapter 1: The Language of Mathematics

Proposition: A sentence that is either true or false (but not both)

Examples:

Propositions

- $2 \times 5 = 10$ (true)
- $\pi = 3$ (false)
- Every even integer greater than two is the sum of two prime numbers

  (Goldbach conjecture from 1742, unknown if true or false)

Not Propositions

- $n^2 + 3n > 7$
- $m < n$
- $12 - 5$

These two are predicates. They become propositions when values are assigned to n (and m for the 2^{nd} one)

Statement : a proposition or predicate

(For example "12 − 5" is not a statement)

Logical Connectives : (or / and / not)

These are "tools" to put together simple statements and form complicated ones

The connective "or" :

Consider two statements P & Q each of which may be either true (T) or false (F).

The statement "P or Q" is also either true or false depending on the truth value of each of P, Q.

In fact, we define the logical connective "or" by means of a truth table :

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

Examples: • $a = 0$ or $b = 0$

• $a \geq 0$ $\quad (a > 0$ or $a = 0)$

• $a = \pm b$ $\quad (a = b$ or $a = -b)$

Caution: The 'connective or' is an "inclusive or" i.e. for the statement 'P or Q' to be true, it is enough for one of the two statements P, Q to be true (if both are true, 'P or Q' is still true - see the truth table)

Example $1 = \pm 1$ is true because $1 = 1$ is true

The connective 'and'

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

Example $3 < \pi < 4$ ($3 < \pi$ and $\pi < 4$)

The connective 'not' (negation of a statement)

P	not P
T	F
F	T

Example: Let P be the statement

'For real numbers a, if $f(a) = 0$, then $a > 0$'

Write down the statement (not P)

Solution: There exists some real number a where $a \leq 0$ such that $f(a) = 0$

We will deal with such statements formally next.

Chapter 2: Implications

Proof: A sequence of statements starting with true statements and finishing with the statement to be proved

Informally, the way we move from statement to statement is via 'implications'

Example: Suppose $P(n)$ is the statement '$n > 2$' and $Q(n)$ is the statement '$n > 0$'. Now, we can define a new statement '$P(n) \Rightarrow Q(n)$'

$n > 2 \Rightarrow n > 0$

hypothesis   conclusion

this is obviously a true statement for each n (while $P(n)$ & $Q(n)$ are not)

	$P(n)$	$Q(n)$
$n \leq 0$	F	F
$n=1$	F	T
$n=2$	F	T
$n>2$	T	T

Notice above:
- when P was true, so was Q
- when P was false, Q could be anythi
- Q was never false, while P was true

Definition:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

'Funny' examples:

- The statement
  $(\pi < 4) \Rightarrow (1+1=2)$ is true

- The statement
  $(\pi < 4) \Rightarrow (1+1=3)$ is false (why?)

Note the absence of a cause/effect relationship in our mathematical use of '$P \Rightarrow Q$'

Universal Implication: An implication where the predicate leads to a true proposition whatever value is assigned to the free variable.

Example: $n > 3 \Rightarrow n > 0$

Example (not a universal statement):

$n \geq 4 \Rightarrow n \geq 6$ is not universally true

	$n \geq 4$	$n \geq 6$	$n \geq 4 \Rightarrow n \geq 6$
$n < 4$	F	F	T
$4 \leq n < 6$	T	F	F
$n \geq 6$	T	T	T

this is where the issue is

There exist values of n for which the hypothesis is true and the conclusion is false.

How to read the statement $P \Rightarrow Q$:

(i) If P then Q
(ii) P implies Q
(iii) Q if P
(iv) P only if Q
(v) Q whenever P
(vi) P is sufficient for Q
(vii) Q is necessary for P

Definition:

$(P \Leftrightarrow Q)$ means $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$

(i) P is equivalent to Q
(ii) P is nec. & suff for Q
(iii) P if & only if Q (P iff Q)
(iv) P precisely when Q

(don't like)

Arithmetic

Some definitions:

Given two integers a and b, we say that b divides a or that a is a multiple of b to mean that there is an integer q with $a = bq$

Examples: 3 divides 6
-4 divides 8
n divides 0 whatever integer n is

(why?)

Given an integer n, we say n is even to mean 2 divides n

Given an integer n, we say n is odd to mean n is not even.

Notice how we used the definition of 'even' to define 'odd', and the definition of 'divides' to define even.

Such chains of definitions are common.

Example: Prove the proposition (1023 is odd)

Proof: We need to show that 1023 is odd, i.e., that it is not even, i.e., that there is no integer q such that $1023 = 2q$.

The issue is we can't go through all integers q and check them one by one.

Instead, we note that there are two case $q \leq 511$ or $q \geq 512$, so either $2q \leq 1022$ or $2q \geq 1024$. So $2q \neq 1023$. So 1023 is odd. $\blacksquare$

end of proof symbol

In the remainder of the course we will assume the following algebraic properties of real numbers, and of the + and × operations.

For two real numbers a & b

(i) Commutativity: $a+b = b+a$

$$ab = ba$$

(ii) Associativity: $a + (b + c) = (a + b) + c$

$$(ab)c = a(bc)$$

(iii) Distributivity: $a(b + c) = ab + ac$

$$(a + b)c = ac + bc$$

(iv) $a + 0 = a = 0 + a$

$$a \times 1 = a = 1 \times a$$

(v) Subtraction: $a + x = 0$ has the unique solution $x = -a$ so

$$a + x = b \iff x = b + (-a) = b - a$$

(vi) Division: If $a \neq 0$, $ax = 1$ has the unique solution $x = 1/a$ so

$$ax = b \iff x = b \cdot \left(\frac{1}{a}\right) = ba^{-1}.$$

All algebraic properties of numbers can be deduced from the above properties.

Chapter 3: Proofs

Many theorems are of the form ($P \Rightarrow Q$)

Recall:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$P \Rightarrow Q$ is true when P is False (regardless of Q)

So to prove ($P \Rightarrow Q$) is true it is enough to assume that P is true and deduce Q

Direct proof

Example For positive real numbers a & b, we have $a < b \Rightarrow a^2 < b^2$

Proof: (By the above reasoning) we may assume that $a < b$. Now,

$$(a < b) \underset{1}{\Rightarrow} (a^2 < ab)$$

and

$$(a < b) \underset{2}{\Rightarrow} (ab < b^2)$$

so $(a < b) \Rightarrow (a^2 < ab) \text{ and } (ab < b^2) \underset{3}{\Rightarrow} a^2 < b^2$.

Hence $(a < b) \Rightarrow (a^2 < b^2)$.

1, 2, 3 are justified by the inequality axioms (also known as order axioms) of real numbers

Axioms: For real numbers a, b, c

(i) Trichotomy: One and only one of the three possibilities $a < b$, $a = b$, $a > b$ is true.

(ii) $a < b \iff a + c < b + c$

(iii) $a < b \iff ac < bc$ if $c > 0$

$a < b \iff ac > bc$ if $c < 0$

(iv) $a < b$ and $b < c \Rightarrow a < c$.

Q: Which of the axioms was used to justify each of 1, 2, 3?

Remark: We could have written the proof with, e.g., more words and fewer symbols, provided we are careful. See book.

Proof by Cases

Example: If $x=2$ or $x=7$ then $x^2-9x+14=0$

Proof: If $x=2$ then $x^2-9x+14=4-18+14=0$
If $x=7$ then $x^2-9x+14=49-63+14=0$.
Hence if $x=2$ or $x=7$ then $x^2-9x+14=0$.

Note: we used the observation that $(P \text{ or } Q) \Rightarrow R$ is equivalent to $(P \Rightarrow R)$ and $(Q \Rightarrow R)$

(check using truth tables)

Example: For real numbers x, $x^2 \geq 0$

Idea (scratch): $x^2 = x \cdot x$

$x=0$, $x>0$, or $x<0$ (trichotomy)

Proof: For a real number x, either $x=0$, $x>0$, or $x<0$ (three cases).

If $x=0$ then $x^2=0$

If $x>0$ then $x^2>0$ (multiplication laws)

If $x<0$ then $x^2>0$.

Hence, $x^2 \geq 0$ as required.

Useful tip: Often, to construct a proof, a good approach is to work backward

Proposition: For real numbers a & b

$$(a<b) \Rightarrow 4ab<(a+b)^2$$

Thought process for constructing proof (scratch)

$$4ab<(a+b)^2 \Leftarrow 4ab<a^2+2ab+b^2$$

actually these are $(\Leftrightarrow)$, but $\Leftarrow$ is enough for us to work backward

$$\Leftarrow 0<a^2-2ab+b^2$$

$$\Leftarrow 0<(a-b)^2$$

$$\Leftarrow a \neq b \text{ (why?)}$$

$\Leftarrow a < b$

so that $a < b \Rightarrow 4ab < (a+b)^2$

Now, we may write a direct proof:

Proof: $a < b \Rightarrow a \neq b \Rightarrow (a-b) \neq 0$

$\Rightarrow 0 < (a-b)^2 \Rightarrow 0 < a^2 - 2ab + b^2$

$\Rightarrow 4ab < a^2 + 2ab + b^2$

$\Rightarrow 4ab < (a+b)^2$

Thus, $a < b \Rightarrow 4ab < (a+b)^2$.